Polarization vector rotation in magnetic fields

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A summary of solution methods for rotation of a polarization vector in magnetic fields with special properties is given. Special attention is given to the so-called resonance-frequency flippers. Some examples are worked out and approximate formula's are derived and compared to numerical simulations.

I. INTRODUCTION

For many applications it is needed that the rotation of the polarization vector, $\vec{P}(t)$ of a particle or other system is accurately known. Here it is assumed that the rotation of the polarization vector can be described by the Larmor equation

$$\frac{d}{dt}\vec{P}(t) = -\gamma\vec{B}(t)\times\vec{P}(t)$$

where the time dependent magnetic field is given by

$$\vec{B}(t) = B(t)\vec{n}(t) \qquad \vec{n}(t) = \begin{pmatrix} \cos\phi(t)\sin\theta(t) \\ \sin\phi(t)\sin\theta(t) \\ \cos\theta(t) \end{pmatrix}$$

and γ is the *gyromagnetic ratio*. For neutrons $\gamma \equiv \mu_n/(\hbar/2) = 1.83247\times 10^8$ 1/Ts. This equation can be recast into the following matrix equation

$$\frac{d}{dt}\vec{P}(t) = \gamma\left(\vec{L}\cdot\vec{B}(t)\right)\vec{P}(t) \tag{1}$$

where $\vec{L}$ is the angular momentum matrix vector with components

$$\widehat{L}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad , \quad \widehat{L}_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \widehat{L}_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which can be rewritten as

$$\frac{d}{dt}\vec{P}(t) = \gamma B(t)\widehat{M}(t)\vec{P}(t)$$

where

$$\widehat{M}(t) = \vec{L}\cdot\vec{n}(t) = \cos\phi(t)\sin\theta(t)\widehat{L}_x + \sin\phi(t)\sin\theta(t)\widehat{L}_y + \cos\theta(t)\widehat{L}_z$$

The matrix $\widehat{M}(t)$ has the properties $\widehat{M}(t)^{\mathrm{T}} = \widehat{M}(t)^3 = -\widehat{M}(t)$, where $\widehat{M}(t)^{\mathrm{T}}$ is the transpose of $\widehat{M}(t)$. These properties ensure that $\left|\vec{P}(t)\right|^2$ is constant under this differential equation. The solution methods followed here have been elaborated in [1]. Unfortunately $\widehat{M}(t)$ has a determinant 0 so that it cannot be inverted. This problem can be solved by using the 2D spinor representation of the Larmor equation (see appendix A).

II. CONSTANT FIELD DIRECTION

If the magnetic field changes in magnitude, but not in direction, the matrix $\widehat{M}$ does not depend on time. In this case one can use the ansatz that the polarization vector has an implicit dependence on time via the function

$$F(t) = \gamma\int_0^t B(x)dx$$

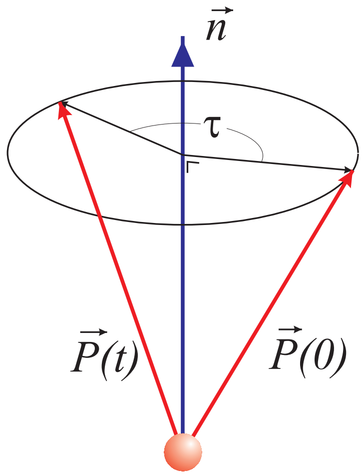

FIG. 1: Effect of rotation matrix on rotation of polarization vector from $\vec{P}(0)$ to $\vec{P}(t)$.

The differential equation for the polarization vector can be written as

$$\frac{d^3}{dF^3}\vec{P}(F) + \frac{d}{dF}\vec{P}(F) = 0$$

with a solution complying with the correct initial conditions

$$\vec{P}(t) = \widehat{R}_{\vec{n}}(F(t))\vec{P}(0)$$

where

$$\widehat{R}_{\vec{n}}(\tau) = \widehat{M}^2(1 - \cos\tau) + \widehat{M}\sin\tau + \widehat{I}$$

is a rotation matrix describing a rotation around axis $\vec{n}$ over an angle τ (see figure 1). $\vec{I}$ is the unit matrix. The precession plane is the plane in which the end point of the polarization is found and is always perpendicular to the rotation axis. When the magnitude of the magnetic field is also constant the solution reduces to

$$\vec{P}(t) = \widehat{R}_{\vec{n}}(\gamma Bt)\vec{P}(0)$$

III. ROTATING FIELD

If the magnetic field changes in magnitude and the direction rotates around a fixed axis it can be described by

$$\vec{B}(t) = B(t)\widehat{R}_{\vec{n}}(\omega t)\vec{m}$$

where $\vec{n}$ is the unit vector in the direction of the axis the magnetic field is rotating around and ω is the rotation frequency. Let us define a rotating polarization vector

$$\vec{P}_r(t) = \widehat{R}_{\vec{n}}(-\omega t)\vec{P}(t)$$

so that eq. (1) becomes

$$\frac{d}{dt}\vec{P}_r(t) = \left(\vec{L} \cdot \{\gamma B(t)\vec{m} - \omega\vec{n}\}\right)\vec{P}_r(t)$$

Note that $\widehat{R}_{\vec{n}}(\tau)^{-1} = \widehat{R}_{\vec{n}}(-\tau)$, $d\widehat{R}_{\vec{n}}(\tau)/d\tau = \left(\vec{L} \cdot \vec{n}\right)\widehat{R}_{\vec{n}}(\tau)$ and $\left(\vec{L} \cdot \widehat{R}_{\vec{n}}(\tau)\vec{m}\right)\widehat{R}_{\vec{n}}(\tau) = \widehat{R}_{\vec{n}}(\tau)\left(\vec{L} \cdot \vec{m}\right)$. This can be written as

$$\frac{d}{dt}\vec{P}_r(t) = \gamma B_e(t)\left(\vec{L} \cdot \vec{v}(t)\right)\vec{P}_r(t)$$

where

$$\gamma B_e(t) = \sqrt{\gamma^2 B(t)^2 + \omega^2 - 2\omega\gamma B(t)(\vec{n}\cdot\vec{m})} \quad \text{and} \quad \vec{v}(t) = \frac{\gamma B(t)\vec{m} - \omega\vec{n}}{\gamma B_e(t)} \tag{2}$$

In this way the new differential equation is completely analogue to the initial eq. (1). As long as $\vec{v}(t)$ does not depend on time, the results can be calculated using the results of the previous section

$$\vec{P}(t) = \widehat{R}_{\vec{n}}(\omega t)\widehat{R}_{\vec{v}}(\gamma B_e t)\vec{P}(0) \tag{3}$$

A. Fast rotating field

For a fast rotating magnetic field, constant in amplitude, eq. (2) can be simplified. Let $\eta = \gamma B/\omega << 1$, then

$$\gamma B_e = \omega\left(1 - (\vec{n}\cdot\vec{m})\eta + (1-(\vec{n}\cdot\vec{m})^2)\frac{\eta^2}{2} + O(\eta^3)\right)$$

and

$$\vec{v} = -\vec{n} + \eta(\vec{m} - (\vec{n}\cdot\vec{m})\vec{n}) + O(\eta^2)\vec{m} + O(\eta^2)\vec{n}$$

so that

$$\widehat{R}_{\vec{v}}(\gamma B_e t) = \widehat{R}_{\vec{n}}(-\gamma B_e t) + O(\eta)\widehat{R}_{\vec{v}}(\gamma B_e t)$$

and

$$\vec{P}(t) \approx \widehat{R}_{\vec{n}}(\omega t - \gamma B_e t)\vec{P}(0)$$

or

$$\vec{P}(t) \approx \widehat{R}_{\vec{n}}\left(\gamma B t(\vec{n}\cdot\vec{m})\right)\widehat{R}_{\vec{n}}\left(\frac{\gamma^2 B^2}{2\omega}((\vec{n}\cdot\vec{m})^2 - 1)t\right)\widehat{R}_{\vec{n}}\left(\frac{\gamma^3 B^3}{2\omega^2}(\vec{n}\cdot\vec{m})((\vec{n}\cdot\vec{m})^2-1)t\right)\vec{P}(0)$$

of which also the last factor can be omitted when $\omega t << \eta^3$. Hence, a fast rotating field can be replaced by a field of a smaller amplitude in the direction of the rotation axis of the magnetic field. Its direction is averaged, so that its amplitude is reduced. If $\vec{n} \perp \vec{m}$ then the reduction is maximal and the first and last term are just unit matrices. The exact results are shown in fig. 2 for $\eta = 0.4$, 0.2 and 0.1. If $\vec{m}//\vec{n}$ only the first term remains and there is no reduction at all.

B. Slow rotating field

For a slow rotating magnetic field eq. (2) can also be simplified. Let $\zeta(t) = \omega/\gamma B(t) << 1$, then

$$\vec{v} = \vec{m} - (\vec{n} - \vec{m}(\vec{n}\cdot\vec{m}))\zeta(t) + O(\zeta(t)^2)\vec{n}$$

so that

$$\frac{d}{dt}\vec{P}_r(t) \approx \gamma B_e(t)\left(\vec{L}\cdot\vec{m}\right)\vec{P}_r(t)$$

which solution is given in the first section

$$\vec{P}_r(t) = \widehat{R}_{\vec{m}}(F_e(t))\vec{P}_r(0)$$

where $F_e(t) = \int_0^t \gamma B_e(x)dx$, so that

$$\vec{P}(t) \approx \widehat{R}_{\vec{n}}(\omega t)\widehat{R}_{\vec{m}}(F_e(t))\vec{P}(0)$$

Hence, a slowly rotating magnetic fields rotates the precession plane of the polarization vector with the same rotation axis as the magnetic field. Hence, the precession plane remains 'locked' with respect to the magnetic field direction. The polarization vector makes many rotations during one rotation of the precession plane. The exact results are shown in fig. 3 for $\zeta = 2$, 5 and 10.

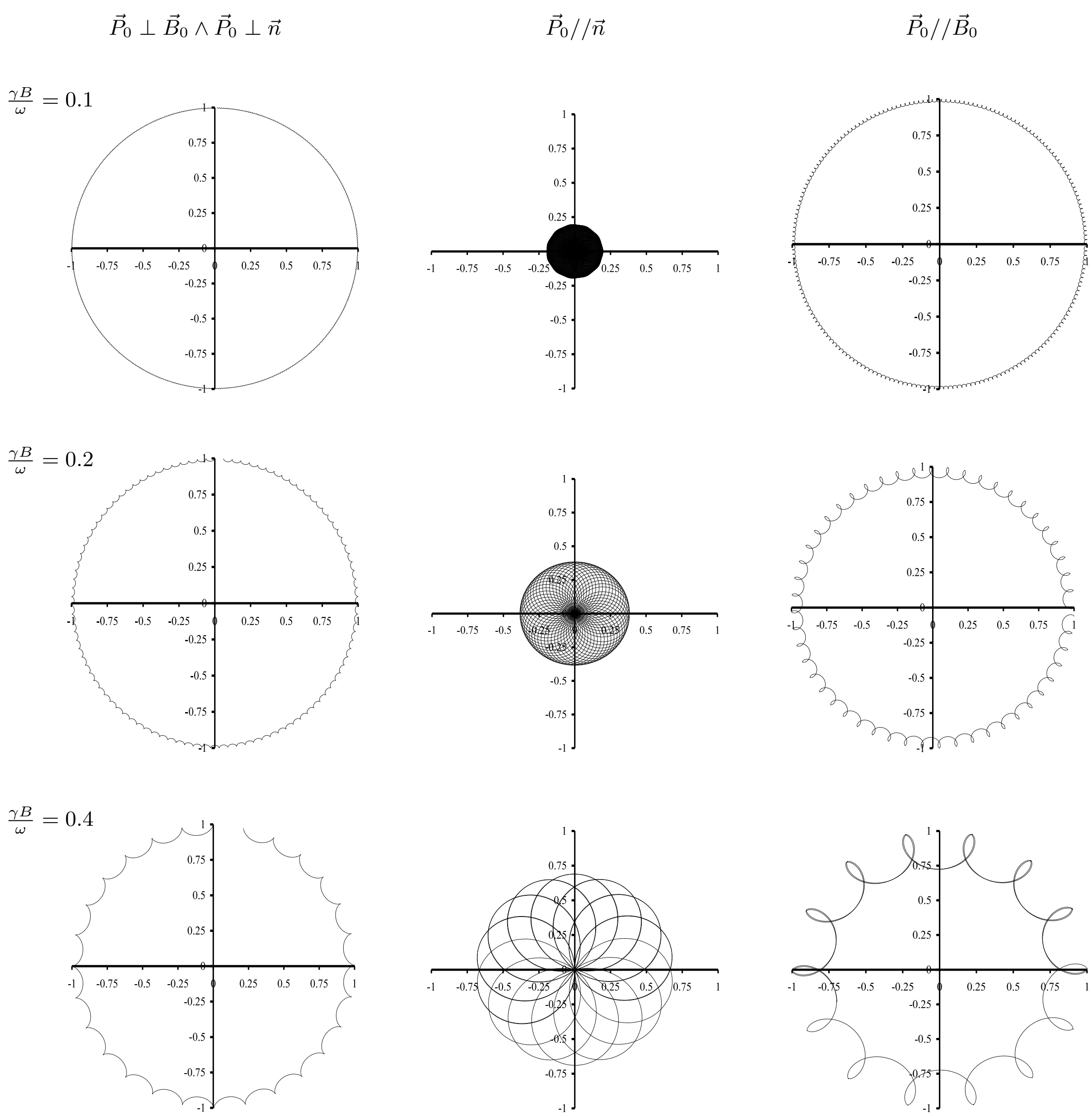

FIG. 2: Rotation of the polarization vector subject to a fast rotating magnetic field (rotation axis in the x-direction and initial magnetic field in the y-direction). Each graph is a polar plot of the y and z components of the polarization vector as function of time. At the left side the initial polarization is perpendicular to both the magnetic field and the rotation axis (z-direction). In the middle the initial polarization is parallel to the rotation axis (x-direction). At the right side the initial polarization is parallel to the initial magnetic field (y-direction). From top to bottom the ratio between magnetic field strength and the rotation frequency is $0.1/\gamma$, $0.2/\gamma$ and $0.4/\gamma$.

C. Resonance flipper

Under certain conditions $\vec{v}$ can be perpendicular to $\vec{n}$. Then, when the initial polarization is equal to $\vec{n}$ a rotation matrix $\widehat{R}_{\vec{v}}(\pi)$ corresponds to a flip the polarization vector. Let $\vec{n}$ be the unit vector in the x-direction and $\vec{v} = (0 \quad \cos\alpha \quad \sin\alpha)^{\mathrm{T}}$, then a flip of the polarization vector occurs for

$$\vec{B} = \frac{\omega}{\gamma}\Big(1 \quad \cos\alpha\sqrt{\eta^2-1} \quad \sin\alpha\sqrt{\eta^2-1}\Big)^{\mathrm{T}} \qquad \text{and} \quad \omega t = \frac{\pi}{\sqrt{\eta^2-1}}$$

where $\eta = \frac{\gamma B}{\omega}$. The field in the x-direction is static and equal to ω/γ, this is the resonance condition. The magnetic field rotates in the y,z-plane with a magnitude of $\sqrt{B^2 - \omega^2/\gamma^2}$. For $\eta >> 1$ the effect on the

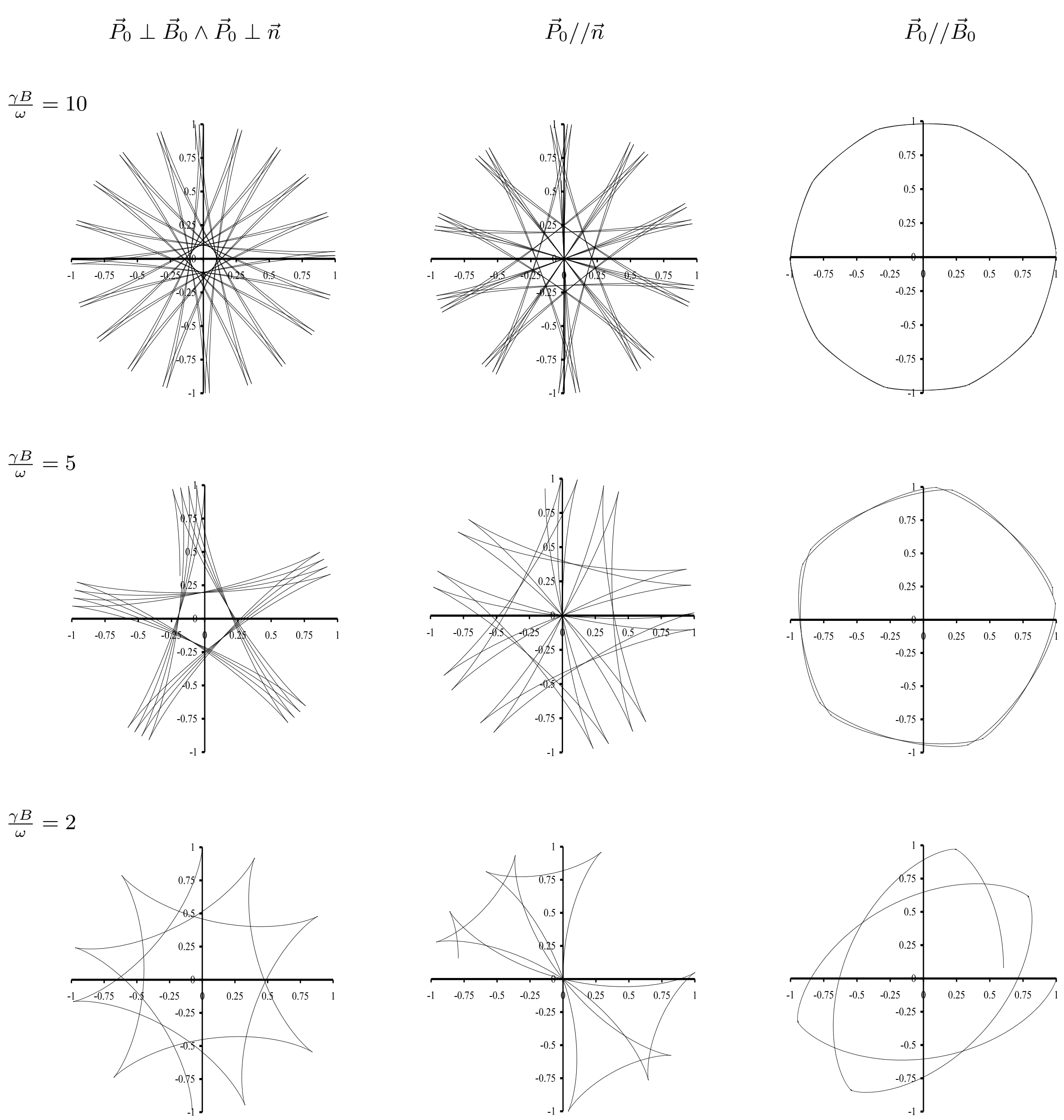

FIG. 3: Rotation of the polarization vector subject to a slow rotating magnetic field (rotation axis in the x-direction and initial magnetic field in the y-direction). Each graph is a polar plot of the y and z components of the polarization vector as function of time. At the left side the initial polarization is perpendicular to both the magnetic field and the rotation axis (z-direction). In the middle the initial polarization is parallel to the rotation axis (x-direction). At the right side the initial polarization is parallel to the initial magnetic field (y-direction). From top to bottom the ratio between magnetic field strength and the rotation frequency is $10/\gamma$, $5/\gamma$ and $2/\gamma$.

polarization vector is a rotation of π due to Larmor precession in a static field $B\vec{v}$. For $\eta = 1 + \epsilon$, where $0 < \epsilon << 1$, this is a rotation of π around an effective field $\omega\sqrt{2\epsilon}/\gamma\vec{v}$. Examples of the resonance flipper for $\alpha = 0$ are shown in fig. 4. The flip does not depend on α, the way the flip occurs however does.

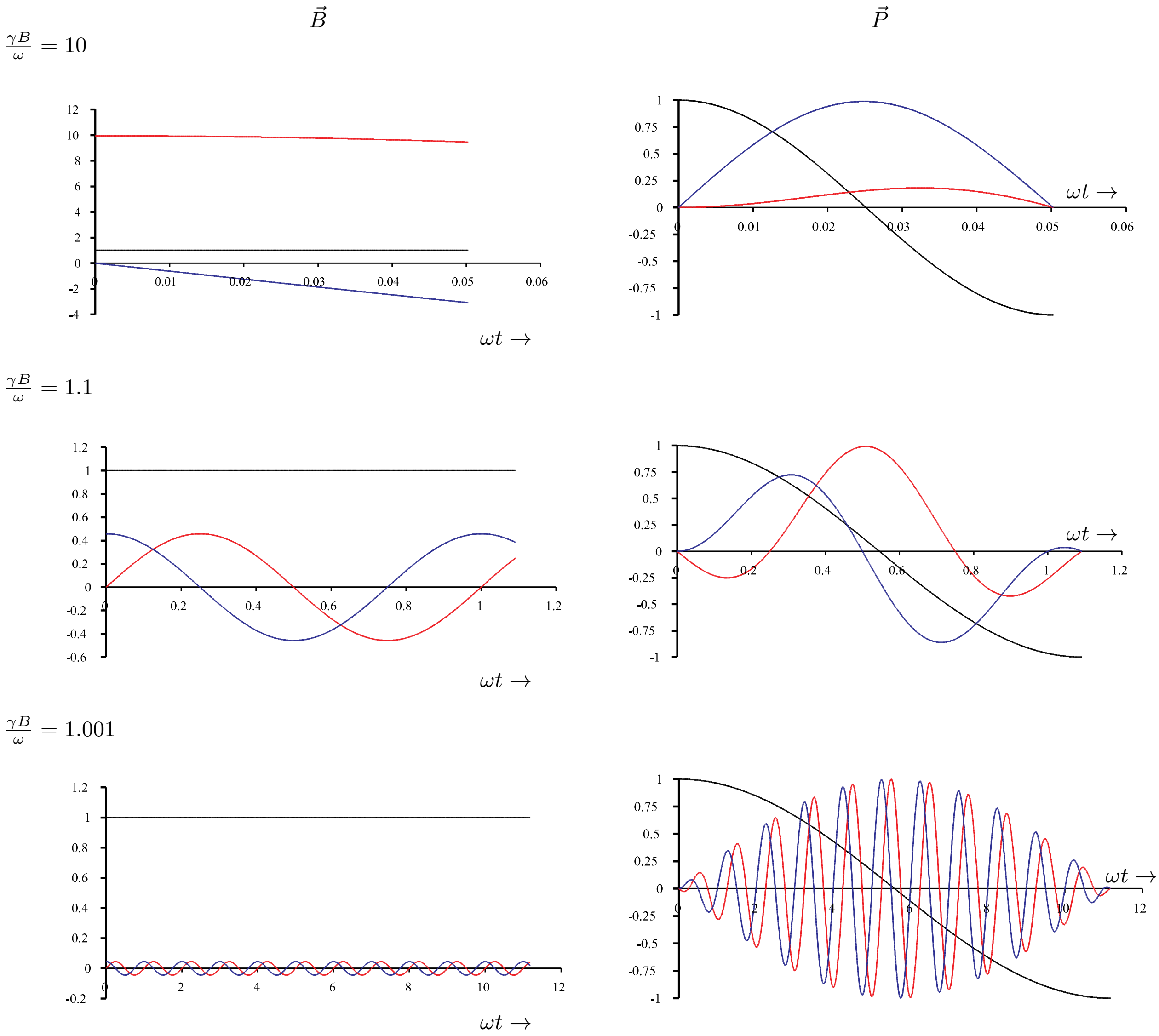

FIG. 4: Flip of the polarization vector subject to a resonant rotating magnetic field (rotation axis in the x-direction and initial magnetic field in the y-direction). Each graph is a plot of the x (black), y (red) and z (blue) components of the magnetic field (left side) and polarization vector (right side) as function of time. From top to bottom the ratio between magnetic field strength and the rotation frequency is $10/\gamma$, $1.1/\gamma$ and $1.001/\gamma$.

IV. OSCILLATING FIELD SUPERIMPOSED ON A STATIC FIELD

An oscillating magnetic field, with frequency ω and magnetic field oscillation amplitude and direction, $2\vec{B}_{osc}$ superimposed on a static magnetic field $\vec{B}_s$ can be described by the superposition of two circularly polarized fields (as described in the previous section), rotating in opposite directions (see figure 5).

$$\vec{B}(t) = \vec{B}_s + \left(\widehat{R}_{\vec{n}}(\omega t) + \widehat{R}_{\vec{n}}(-\omega t)\right)\vec{B}_{osc} \tag{4}$$

where $\vec{n}$ is a unit vector perpendicular to $\vec{B}_{osc}$. This does not define $\vec{n}$ completely. Let us define a rotating polarization vector

$$\vec{P}_r(t) = \widehat{R}_{\vec{n}}(-\omega t)\vec{P}(t)$$

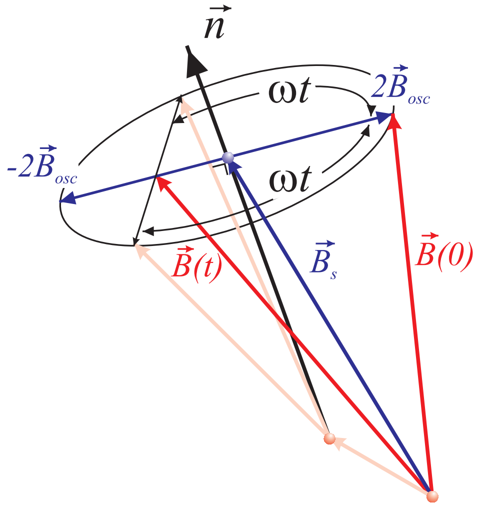

FIG. 5: Oscillating field superimposed on a static field composed from two circularly polarized fields.

so that eq. (1) becomes

$$\frac{d}{dt}\vec{P}_r(t) = \left(\vec{L}\cdot\left\{\gamma\widehat{R}_{\vec{n}}(-\omega t)\vec{B}_s - \omega\vec{n} + \gamma\vec{B}_{osc} + \gamma\widehat{R}_{\vec{n}}(-2\omega t)\vec{B}_{osc}\right\}\right)\vec{P}_r(t)$$

In the special case that $\vec{B}_s // \vec{n}$ (which can always be realized as long as $\vec{B}_s \perp \vec{B}_{osc}$) this can be reduced to

$$\frac{d}{dt}\vec{P}_r(t) = \left(\vec{L}\cdot\left\{\gamma B_e\vec{m} + \gamma\widehat{R}_{\vec{n}}(-2\omega t)\vec{B}_{osc}\right\}\right)\vec{P}_r(t)$$

where

$$\gamma B_e = \sqrt{(\gamma B_s - \omega)^2 + (\gamma B_{osc})^2}$$

and

$$\vec{m} = \frac{(\gamma B_s - \omega)\vec{n} + \gamma\vec{B}_{osc}}{\gamma B_e}$$

This differential equation can be simplified further by introducing

$$\vec{P}_{rr}(t) = \widehat{R}_{\vec{m}}(-\gamma B_e t)\vec{P}_r(t)$$

so that

$$\frac{d}{dt}\vec{P}_{rr}(t) = \gamma\left(\vec{L}\cdot\left\{\widehat{R}_{\vec{m}}(-\gamma B_e t)\widehat{R}_{\vec{n}}(-2\omega t)\vec{B}_{osc}\right\}\right)\vec{P}_{rr}(t)$$

Now, if we assume that $\omega >> \gamma B_{osc}$, then the field $\widehat{R}_{\vec{n}}(-2\omega t)\vec{B}_{osc}$ can be replaced by its effective value as derived in the previous section, so that

$$\frac{d}{dt}\vec{P}_{rr}(t) = \gamma\left(\vec{L}\cdot\left\{\frac{\gamma B_{osc}^2}{4\omega}\widehat{R}_{\vec{m}}(-\gamma B_e t)\vec{n}\right\}\right)\vec{P}_{rr}(t)$$

which can be solved by returning to $\vec{P}_r(t)$

$$\vec{P}_r(t) = \widehat{R}_{\vec{m}}(\gamma B_e t)\vec{P}_{rr}(t)$$

so that

$$\frac{d}{dt}\vec{P}_r(t) = \left(\vec{L}\cdot\left\{\left(\frac{\gamma^2 B_{osc}^2}{4\omega} + \gamma B_s - \omega\right)\vec{n} + \gamma\vec{B}_{osc}\right\}\right)\vec{P}_r(t)$$

which can be rewritten as

$$\frac{d}{dt}\vec{P}_r(t) = \gamma\left(\vec{L}\cdot B_v\vec{v}\right)\vec{P}_r(t)$$

with

$$\gamma B_v = \sqrt{\left(\frac{\gamma^2 B_{osc}^2}{4\omega} + \gamma B_s - \omega\right)^2 + (\gamma B_{osc})^2}$$

and

$$\vec{v} = \frac{\left(\frac{\gamma^2 B_{osc}^2}{4\omega} + \gamma B_s - \omega\right)\vec{n} + \gamma\vec{B}_{osc}}{\gamma B_v}$$

The solution reads

$$\vec{P}_r(t) = \widehat{R}_{\vec{v}}(\gamma B_v t)\vec{P}_r(0)$$

and

$$\vec{P}(t) = \widehat{R}_{\vec{n}}(\omega t)\widehat{R}_{\vec{v}}(\gamma B_v t)\vec{P}(0)$$

The same equation as eq. (3). Hence, under the two conditions $\vec{B}_s \perp \vec{B}_{osc}$ and $\omega >> \gamma B_{osc}$, a linear oscillating magnetic field has the same effect on the rotation of the polarization vector as a circularly polarized magnetic field rotation.

A. Resonance flipper

Similar to the resonance flip for a rotating field, a resonance flip for an oscillating field can be constructed. Again $\vec{v}$ should be perpendicular to $\vec{n}$. Then, when the initial polarization is equal to $\vec{n}$ a rotation matrix $\widehat{R}_{\vec{v}}(\pi)$ corresponds to a flip of the polarization vector. Let $\vec{n}$ be the unit vector in the x-direction and $\vec{v} = (0\ \ 1\ \ 0)^{\mathrm{T}}$, then a flip occurs for

$$\vec{B}_s = (\omega/\gamma - \gamma B_{osc}^2/4\omega)\vec{n} \quad \text{and} \quad \omega t = \frac{\pi\omega}{\gamma B_{osc}}$$

As $\omega >> \gamma B_{osc}$, this resonance condition corresponds to $\gamma B_s \approx \omega$. The magnetic field oscillates in the y-direction. Examples of the resonance flipper are shown in fig. 6. For large $\gamma\omega/B_{osc}$, the results is comparable to the resonance flip for a rotating field. For small values of this parameter the flip is incomplete.

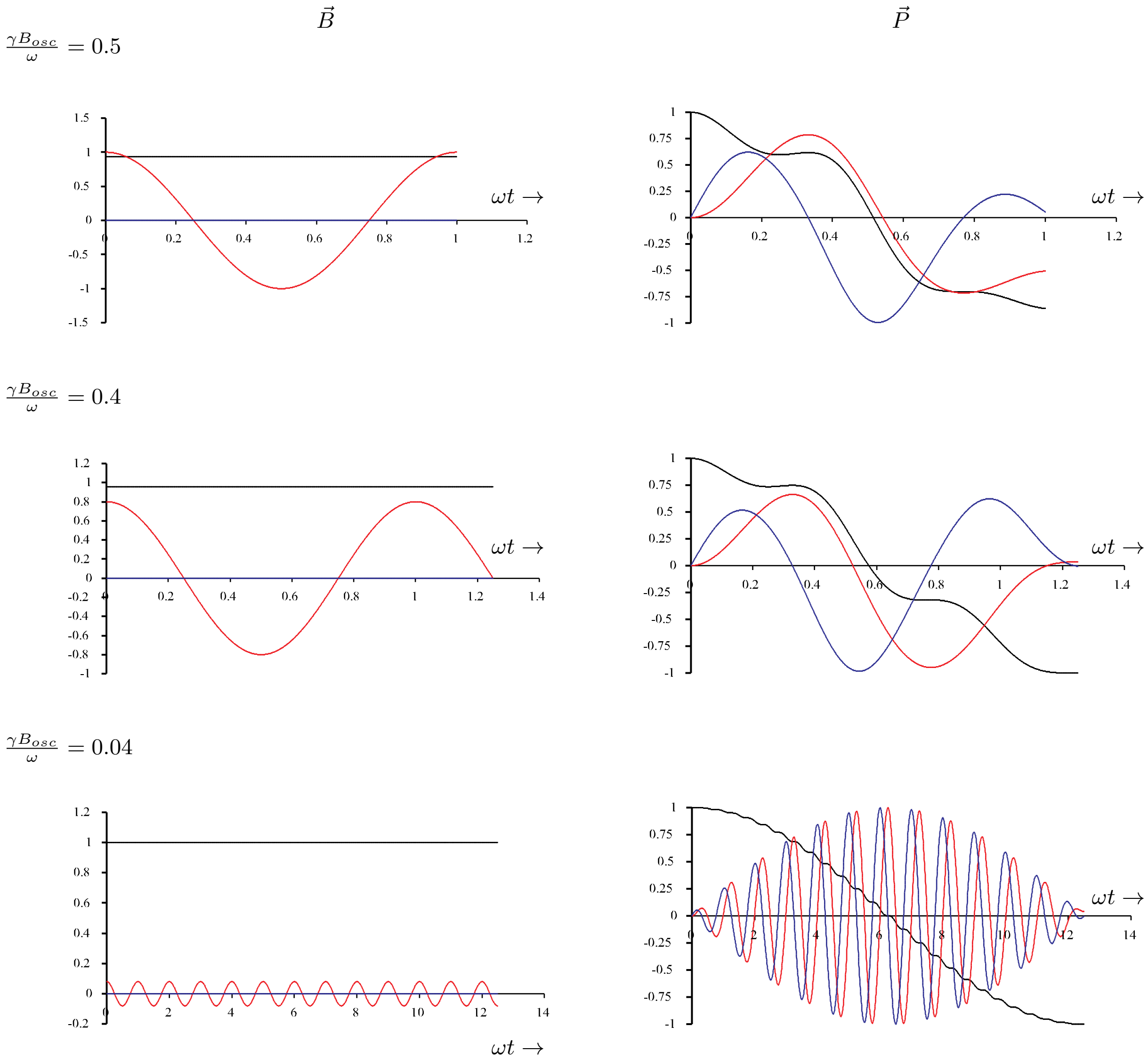

FIG. 6: Flip of the polarization vector subject to a resonant oscillating magnetic field (static field in the x-direction and oscillating direction in the y-direction). Each graph is a plot of the x (black), y (red) and z (blue) components of the magnetic field (left side) and polarization vector (right side) as function of time. From top to bottom the ratio between oscillating magnetic field strength and the oscillating frequency is $0.5/\gamma$, $0.4/\gamma$ and $0.04/\gamma$.

V. ADIABATIC RESONANCE-FREQUENCY FLIPPER IN TIME

Let us consider the field

$$\vec{B}(t) = B_s \vec{n}_{gr} + B_{gr} \cos(\omega_{gr} t) \vec{n}_{gr} + 2 B_{rf} \cos(\omega_{rf} t + \phi_{rf}) \sin(\omega_{gr} t) \vec{n}_{rf} \tag{5}$$

where $\vec{n}_{gr} \cdot \vec{n}_{rf} = 0$. To find the solution for the polarization rotation, let us define a rotating polarization vector

$$\vec{P}_r(t) = \widehat{R}_{\vec{n}_{gr}}(-\omega_{rf} t - \phi_{rf}) \vec{P}(t)$$

so that eq. (1) becomes

$$\frac{d}{dt} \vec{P}_r(t) = \gamma \left(\vec{L} \cdot \vec{B}_r(t) \right) \vec{P}_r(t)$$

where

$$\vec{B}_r(t) = (B_s + B_{gr}\cos(\omega_{gr}t) - \omega_{rf}/\gamma)\,\vec{n}_{gr} + 2B_{rf}\cos(\omega_{rf}t + \phi_{rf})\sin(\omega_{gr}t)\widehat{R}_{\vec{n}_{gr}}(-\omega_{rf}t - \phi_{rf})\vec{n}_{rf}$$

Using the fact that $2\cos\tau\vec{n} = (\widehat{R}_{\vec{n}_\perp}(\tau) + \widehat{R}_{\vec{n}_\perp}(-\tau))\vec{n}$ when $\vec{n}\cdot\vec{n}_\perp = 0$ this can be rewritten as

$$\vec{B}_r(t) = (B_s - \omega_{rf}/\gamma)\,\vec{n}_{gr} + B_{rf}\sin(\omega_{gr}t)\widehat{R}_{\vec{n}_{gr}}(-2\omega_{rf}t - 2\phi_{rf})\vec{n}_{rf} + \tag{6}$$

$$B_e\left(\widehat{R}_{\vec{n}_\perp}(\omega_{gr}t)\cos(\alpha_e - \pi/4) + \widehat{R}_{\vec{n}_\perp}(-\omega_{gr}t)\sin(\pi/4 - \alpha_e)\right)\vec{n}_{gr}$$

where

$$B_e = \sqrt{\frac{B_{gr}^2 + B_{rf}^2}{2}} \quad , \quad \tan\alpha_e = \frac{B_{rf}}{B_{gr}}$$

and $\vec{n}_\perp\cdot\vec{n}_{rf} = \vec{n}_\perp\cdot\vec{n}_{gr} = 0$ so that $\vec{n}_{rf} = \widehat{R}_{\vec{n}_\perp}(\pi/2)\vec{n}_{gr}$. The first term of $\vec{B}_r(t)$ is zero when $\omega_{rf} = \gamma B_s$ and the last term becomes a simple rotation when $\alpha_e = \pi/4$, or equivalently $B_{rf} = B_{gr} = B_e$. Then the solution can be found by introducing

$$\vec{P}_{rr}(t) = \widehat{R}_{\vec{n}_\perp}(\omega_{gr}t)\vec{P}_r(t)$$

so that the differential equation becomes

$$\frac{d}{dt}\vec{P}_{rr}(t) = \gamma\left(\vec{L}\cdot\vec{B}_{rr}(t)\right)\vec{P}_{rr}(t)$$

where

$$\vec{B}_{rr}(t) = \frac{\omega_{gr}}{\gamma}\vec{n}_\perp + B_{rf}\vec{n}_{gr} + \widehat{R}_{\vec{n}_\perp}(\omega_{gr}t)\widehat{R}_{\vec{n}_{gr}}(-2\omega_{rf}t)B_{rf}\sin(\omega_{gr}t)\widehat{R}_{\vec{n}_{gr}}(-2\phi_{rf})\vec{n}_{rf}$$

The last term is a fast rotating field that can be replaced by its effective value as was done in the previous section, reducing the magnetic field to

$$\vec{B}_{rr}(t) = \frac{\omega_{gr}}{\gamma}\vec{n}_\perp + B_{rf}\vec{n}_{gr} - \widehat{R}_{\vec{n}_\perp}(\omega_{gr}t)\frac{\gamma B_{rf}^2\sin(\omega_{gr}t)^2}{4\omega_{rf}}\vec{n}_{gr}$$

The final step is to take $B_{rf} << B_s$ so that the last term can be ignored and $\vec{B}_{rr}(t)$ is a static field, that can be solved according to the method in section II. The result is

$$\vec{P}_{rr}(t) = \widehat{R}_{\vec{v}}(\Omega t)\vec{P}_{rr}(0)$$

where

$$\Omega = \sqrt{\omega_{gr}^2 + \gamma^2 B_{rf}^2} \qquad \text{and} \qquad \vec{v} = \frac{\omega_{gr}\vec{n}_\perp + \gamma B_{rf}\vec{n}_{gr}}{\Omega}$$

so that

$$\vec{P}(t) = \widehat{R}_{\vec{n}_{gr}}(\omega_{rf}t + \phi_{rf})\widehat{R}_{\vec{n}_\perp}(-\omega_{gr}t)\widehat{R}_{\vec{n}_{rf}}(\beta)\widehat{R}_{\vec{n}_{gr}}(\Omega t)\widehat{R}_{\vec{n}_{rf}}(-\beta)\widehat{R}_{\vec{n}_{gr}}(-\phi_{rf})\vec{P}(0)$$

where it was used that

$$\widehat{R}_{\vec{v}}(\Omega t) = \widehat{R}_{\vec{n}_{rf}}(\beta)\widehat{R}_{\vec{n}_{gr}}(\Omega t)\widehat{R}_{\vec{n}_{rf}}(-\beta)$$

when

$$\sin\beta = \frac{\omega_{gr}}{\Omega} \qquad \text{and} \qquad \vec{v} = \widehat{R}_{\vec{n}_{rf}}(\beta)\vec{n}_{gr}$$

In the limit that $\beta \to 0$ (adiabatic rotation) and $\omega_{gr}t = \pi$ the rotation is a flip when the initial polarization was parallel to $\vec{n}_{gr}$

$$\vec{P}(t) = \widehat{R}_{\vec{n}_{gr}}(\omega_{rf}t - \Omega t + 2\phi_{rf})\widehat{R}_{\vec{n}_\perp}(\pi)\vec{P}(0)$$

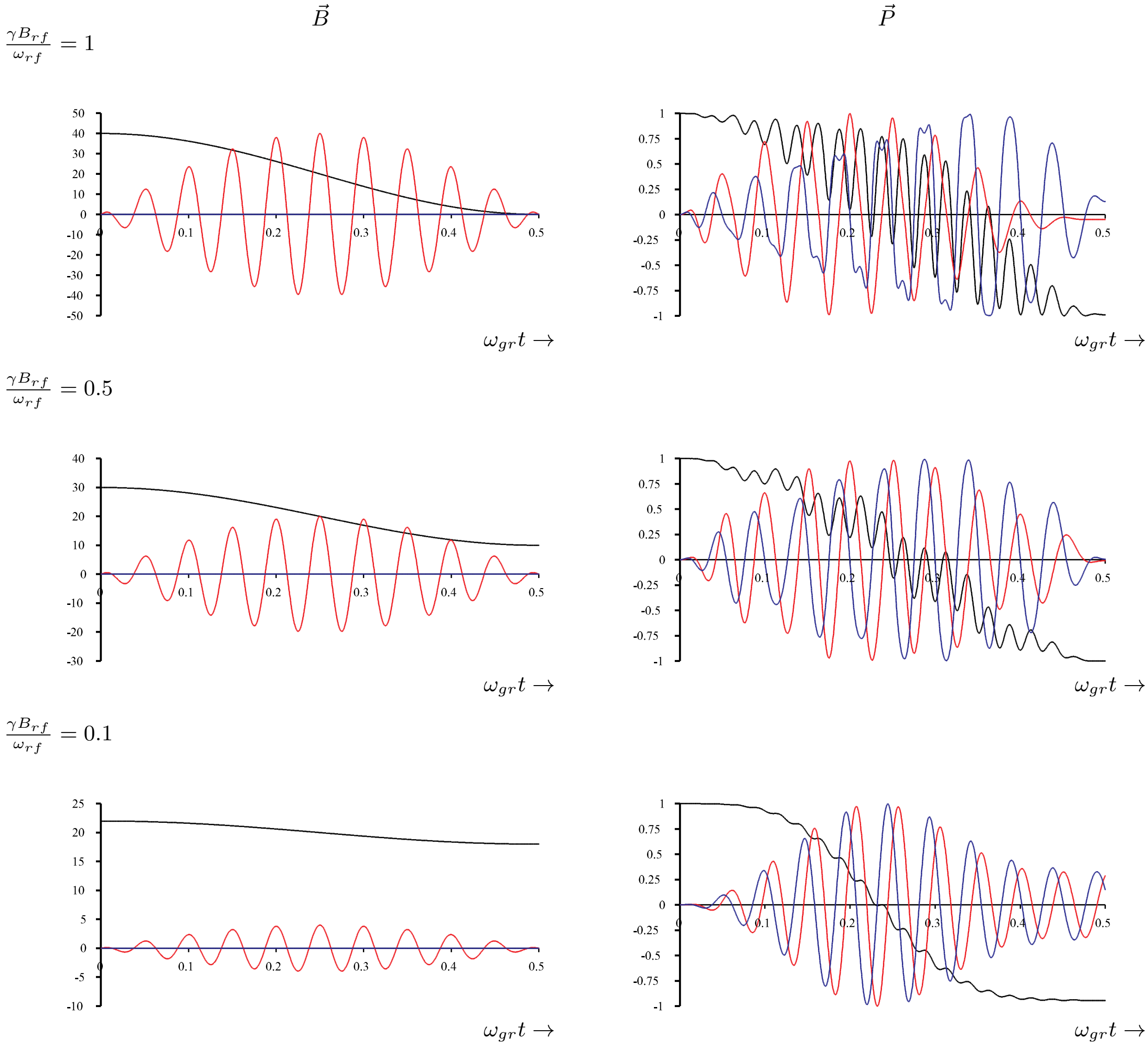

FIG. 7: Flip of the polarization vector subject to magnetic field corresponding to eq. 5 (static field in the x-direction with a magnitude equal to the resonance condition ($B_s = \gamma\omega_{rf}$) and oscillating direction in the y-direction with $B_{gr} = B_{rf}$. $\omega_{rf} = 20\omega_{gr}$ Each graph is a plot of the x (black), y (red) and z (blue) components of the magnetic field (left side) and polarization vector (right side) as function of time. From top to bottom the ratio between oscillating magnetic field strength and the oscillating frequency is $1/\gamma$, $0.5/\gamma$ and $0.1/\gamma$.

where it was used that

$$\widehat{R}_{\vec{n}_\perp}(\pi)\widehat{R}_{\vec{n}_{gr}}(\tau) = \widehat{R}_{\vec{n}_{gr}}(-\tau)\widehat{R}_{\vec{n}_\perp}(\pi)$$

Several examples are shown in fig. 7. Differently, when $\vec{P}(0)//\vec{n}_\perp$, then there is no flip, but a rotation around $\vec{n}_{gr}$ or when $\vec{P}(0)//\vec{n}_{rf}$, then there is a flip and a rotation around $\vec{n}_{gr}$. Further, after double the time $\omega_{gr}t = 2\pi$ the rotation is

$$\vec{P}(t) = \widehat{R}_{\vec{n}_{gr}}(\omega_{rf}t + \Omega t)\vec{P}(0)$$

independent of the phase of $\vec{B}_{rf}$.

VI. ADIABATIC RESONANCE-FREQUENCY FLIPPER IN SPACE

The gradient field of the previous section can be realized by a field constant in time when the neutron moves through it

$$\vec{B}_{gr}(\vec{x}) = u(\vec{x}\cdot\vec{k}_{gr})u(\pi - \vec{x}\cdot\vec{k}_{gr})B_{gr}\cos(\vec{x}\cdot\vec{k}_{gr})\vec{n}_{gr}$$

where $u(x)$ is the Heaviside step function

$$x < 0 \;:\; u(x) = 0 \quad \text{and} \quad x \geq 0 \;:\; u(x) = 1$$

Let $\vec{x}_n(t) = \vec{v}_n t$ be the position of the neutron at time t, then

$$\vec{B}_{gr}(t) = u(\omega_{gr}t)u(\pi - \omega_{gr}t)B_{gr}\cos(\omega_{gr}t)\vec{n}_{gr}$$

where $\omega_{gr} = \vec{v}_n \cdot \vec{k}_{gr}$. Similarly, the oscillating field can be constructed from a field oscillating in time

$$\vec{B}_{rf}(\vec{x},t) = 2B_{rf}u(\vec{x}\cdot\vec{k}_{gr})u(\pi - \vec{x}\cdot\vec{k}_{gr})\sin(\vec{x}\cdot\vec{k}_{gr})\cos(\omega_{rf}t + \phi_{rf})\vec{n}_{rf}$$

so that

$$\vec{B}_{rf}(t) = 2B_{rf}u(\omega_{gr}t)u(\pi - \omega_{gr}t)\sin(\omega_{gr}t)\cos(\omega_{rf}t + \phi_{rf})\vec{n}_{rf}$$

The rotation of the polarization vector of the neutron passing these fields is given in the previous section

$$\vec{P}_1 = \widehat{R}_{\vec{n}_{gr}}\left(\frac{(\omega_{rf} - \Omega)\pi}{\omega_{gr}} + 2\phi_{rf}\right)\widehat{R}_{\vec{n}_\perp}(\pi)\vec{P}_0$$

Now, let us assume that a similar field pattern is created at a location $\vec{v}_n T$ downstream of the neutrons path, then the gradient field is

$$\vec{B}_{gr}(\vec{x}) = u((\vec{x} - \vec{v}_n T)\cdot\vec{k}_{gr})u(\pi - (\vec{x} - \vec{v}_n T)\cdot\vec{k}_{gr})B_{gr}\cos((\vec{x} - \vec{v}_n T)\cdot\vec{k}_{gr})\vec{n}_{gr}$$

so that

$$\vec{B}_{gr}(\vec{x}) = u(\omega_{gr}(t - T))u(\pi - \omega_{gr}(t - T))B_{gr}\cos(\omega_{gr}(t - T))\vec{n}_{gr}$$

Similarly, the oscillating field is

$$\vec{B}_{rf}(\vec{x},t) = 2B_{rf}u((\vec{x} - \vec{v}_n T)\cdot\vec{k}_{gr})u(\pi - (\vec{x} - \vec{v}_n T)\cdot\vec{k}_{gr})\sin((\vec{x} - \vec{v}_n T)\cdot\vec{k}_{gr})\cos(\omega_{rf}t + \phi_{rf})\vec{n}_{rf}$$

where we assumed that the oscillating fields are driven with exactly the same phase, so that

$$\vec{B}_{rf}(t) = 2B_{rf}u(\omega_{gr}(t - T))u(\pi - \omega_{gr}(t - T))\sin(\omega_{gr}(t - T))\cos(\omega_{rf}t + \phi_{rf})\vec{n}_{rf}$$

The rotation of the polarization vector of the neutron passing these fields is given in the previous section where now an adapted phase for $\vec{B}_{rf}$ needs to be used

$$\vec{P} = \widehat{R}_{\vec{n}_{gr}}\left(\frac{(\omega_{rf} - \Omega)\pi}{\omega_{gr}} + 2\phi_{rf} + 2\omega_{rf}T\right)\widehat{R}_{\vec{n}_\perp}(\pi)\vec{P}_1$$

After both fields the polarization rotation has become

$$\vec{P} = \widehat{R}_{\vec{n}_{gr}}(-2\omega_{rf}T)\vec{P}_0$$

comparable to a polarization rotation as if the neutron moves over a distance $\vec{v}_n T$ through an homogenous magnetic field of magnitude $2B_s$.

VII. CONCLUSIONS

Solutions for rotation of a polarization vector in magnetic fields can be derived from the Larmor equations. Several approximations for slow and fast rotating or oscillating fields give useful insight in the behavior of the polarization vector. This insight can be used to manipulate the polarization vector. Examples for resonant flippers indicated the possibilities of the method.

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[1] W. H. Kraan, *Instrumentation to handle thermal polarized neutron beams* (Delft Universtiy of Technology, 2004), 1st ed., phD thesis.

APPENDIX A: EQUIVALENT SPINOR DESCRIPTION

The spinor description of the Larmor equation reads

$$\frac{d}{dt}\vec{S}(t) = \frac{1}{2}\gamma\, i\left(\vec{\sigma}\cdot\vec{B}(t)\right)\vec{S}(t) \tag{A1}$$

where $\vec{S}(t)$ is the complex spinor and $\vec{\sigma}$ is the Pauli spin matrix vector with components

$$\widehat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \widehat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \widehat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The polarization vector is given by

$$\vec{P}(t) = \frac{\vec{S}(t)^\dagger \vec{\sigma}\vec{S}(t)}{\left|\vec{S}(t)\right|^2}$$

where $\vec{S}(t)^\dagger$ is the transpose and complex conjugate of $\vec{S}(t)$. Eq. (A1) can be written in full matrix form

$$\frac{d}{dt}\vec{S}(t) = \frac{\gamma B(t)}{2} i\widehat{N}(t)\vec{S}(t) \tag{A2}$$

where

$$\widehat{N}(t) = \vec{\sigma}\cdot\vec{n}(t) = \cos\phi(t)\sin\theta(t)\widehat{\sigma}_x + \sin\phi(t)\sin\theta(t)\widehat{\sigma}_y + \cos\theta(t)\widehat{\sigma}_z$$

Now $\widehat{N}(t)^2 = \widehat{I}$, the identity matrix and the determinant of $\widehat{N}(t)$ is -1, so that $\widehat{N}(t)^{-1} = \widehat{N}(t)$. Further $\widehat{N}(t)^\dagger = \widehat{N}(t)$, so that it can be shown that $\left|\vec{S}(t)\right|^2$ is constant under this differential equation. Eq. (A2) can be inverted to

$$\vec{S}(t) = -\frac{2}{\gamma B(t)} i\widehat{N}(t)\frac{d}{dt}\vec{S}(t) \tag{A3}$$

The correspondence between the 3D polarization description and 2D spinor description is fundamental and complete. Solutions found using one description can always be represented in the other. For constant $\widehat{N}$ eq. (A2) can be solved in a similar manner as for the 3D-polarization description, taking

$$G(t) = \frac{\gamma}{2}\int_0^t B(x)dx$$

reduces the differential equation to

$$\frac{d^2}{dG^2}\vec{S}(G) + \vec{S}(G) = 0$$

with a solution complying with the correct initial conditions

$$\vec{S}(t) = \left(\widehat{I}\cos G(t) + i\,\widehat{N}\sin G(t)\right)\vec{S}(0)$$